

TEICHMÜLLER SPACES OF RIEMANN SURFACES WITH ORBIFOLD POINTS OF ARBITRARY ORDER AND CLUSTER VARIABLES

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ABSTRACT. We generalize a new class of cluster type mutations for which exchange transformations are given by reciprocal polynomials. In the case of second-order polynomials of the form $x + 2 \cos \pi/n_o + x^{-1}$ these transformations are related to triangulations of Riemann surfaces of arbitrary genus with at least one hole/puncture and with an arbitrary number of orbifold points of arbitrary integer orders n_o . We propose the dual graph description of the corresponding Teichmüller spaces, construct the Poisson algebra of the Teichmüller space coordinates, propose the combinatorial description of the corresponding geodesic functions and find the mapping class group transformations.

1. INTRODUCTION

Since their appearance, cluster variables [14] find applications in geometry. An important example of the cluster variables in the case where exchange polynomial is of the second-order is provided by λ -lengths [23], [22] of curves that make ideal triangle partitions of fundamental domains of Riemann surfaces with punctures. These systems were generalized in [8], [9] to the case of Riemann surfaces with holes. At the same time, a combinatorial description of geodesic functions in terms of the dual variables, the shear coordinates, as well as their quantization, was developed in [5]. Amazingly enough, transition from punctures to holes does not effectively change the corresponding cluster algebra. Generalizations of Teichmüller spaces of Riemann surfaces to the case of bordered Riemann surfaces [20] or ciliated Riemann surfaces [10] were constructed. The corresponding cluster algebras were developed in [19, 11, 12], whereas the geometrical pattern underlying the bordered Riemann surfaces was identified with that of Riemann surfaces with \mathbb{Z}_2 -orbifold points in [2], [3], where the corresponding mutations (flips) in terms of the shear coordinates were constructed. These flips preserve the sets of geodesic functions; on the level of cluster variables the corresponding transformations were considered in [13]; the mutations of cluster variables were there again given by the standard two-term relations. In [6], the description of Teichmüller spaces of Riemann surfaces with holes and with orbifold points of order two and three was given.

It was, however, an ambiguity in identifying a boundary component with an orbifold point; in the present paper, we demonstrate that this ambiguity is due to choice of the type of an orbifold point corresponding to the given boundary component; we therefore provide the combinatorial description of Riemann surfaces

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with holes and with orbifold points of arbitrary orders. We show that mutations for orbifold points of order greater than two are given by three-, not two-term transformations determined by a second-order reciprocal polynomial. We prove the Laurent phenomenon and positivity property for these transformations. The positive coefficients of Laurent polynomials however become not necessarily integers in the presence of orbifold points of order greater than three. On the shear-coordinate side, we define the complete set of real-valued coordinates, construct all the geodesic functions for such surfaces, all the mapping-class-group transformations, and prove the regularity condition, that is, that all elements of the corresponding Fuchsian group are hyperbolic or parabolic ones except precisely elements conjugate to passing around orbifold points. We therefore have a regular (up to exactly the indicated orbifold points) Riemann surface with holes for any choice of the introduced real coordinates, and vice versa; these coordinates parameterize therefore the corresponding Teichmüller spaces of Riemann surfaces with holes and with orbifold points.

The mutations of orbifold triangulations are particular cases of more general construction. In [18] some generalization of cluster transformations were described that preserve Poisson bracket and have some universal properties. Until recently no applications of these transformations were found. We were happy to see manifestation of our formulas in mutations of orbifold triangulations. Motivated by that we propose a new algebraic construction of generalized cluster algebras with mutations given by reciprocal polynomials. Using the tools of the standard cluster algebra [15], [16], we prove that the Laurent phenomenon holds true in this case as well. For algebras of order greater than two, we do not know whether the positivity property holds in general; it however holds in all tested examples, so we formulate it as a conjecture. We also prove that generalized cluster algebras of finite type satisfy the same Cartan–Killing classification as the standard cluster algebras.

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2. GENERALIZED CLUSTER ALGEBRA

We briefly remind the definition of cluster algebra.

An integer $n \times n$ matrix B is called *skew-symmetrizable* if there exists an integer diagonal $n \times n$ matrix $D = \text{diag}(d_1, \dots, d_n)$, such that the product BD is a skew-symmetric matrix, i.e., $b_{ij}d_j = -b_{ji}d_i$.

Let \mathbb{P} be a *semifield* equipped with commutative multiplication \cdot and addition \oplus . We assume that the multiplicative group of \mathbb{P} is a free abelian group. \mathbb{P} is a *coefficient group* of cluster algebra. $\mathbb{Z}\mathbb{P}$ is the integer group ring, \mathcal{F} is a field of rational functions in n independent variables with coefficients in the field of fractions of $\mathbb{Z}\mathbb{P}$. \mathcal{F} is called an ambient field.

Definition 2.1. A *seed* is a triple $(\mathbf{x}, \mathbf{p}, B)$, where

- $\mathbf{p} = (p_x^\pm)_{x \in \mathbf{x}}$, a $2n$ -tuple of elements of \mathbb{P} is a *coefficient tuple* of cluster \mathbf{x} ;
- $\mathbf{x} = \{x_1, \dots, x_n\}$ is a collection of algebraically independent rational functions of n variables which generates \mathcal{F} over the field of fractions of $\mathbb{Z}\mathbb{P}$;
- B is a skew-symmetrizable *exchange matrix*.

The part \mathbf{x} of seed $(\mathbf{x}, \mathbf{p}, B)$ is called *cluster*, elements $x_i \in \mathbf{x}$ are called *cluster variables*, and B is called *exchange matrix*.

Definition 2.2 (seed mutation). For any k , $1 \leq k \leq n$ we define the *mutation* of seed $(\mathbf{x}, \mathbf{p}, B)$ in direction k as a new seed $(\mathbf{x}', \mathbf{p}', B')$ in the following way:

$$(2.1) \quad b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

$$(2.2) \quad x'_i = \begin{cases} x_i, & \text{if } i \neq k; \\ \frac{p_k^+ \prod_{b_{jk} > 0} x_j^{b_{jk}} + p_k^- \prod_{b_{jk} < 0} x_j^{-b_{jk}}}{x_k}, & \text{otherwise.} \end{cases}$$

$$(2.3) \quad p_k'^{\pm} = p_k^{\mp}$$

$$(2.4) \quad \text{for } i \neq k \quad p_i'^+ / p_i'^- = \begin{cases} (p_k^+)^{b_{ki}} p_i^+ / p_i^-, & \text{if } b_{ki} \geq 0; \\ (p_k^-)^{b_{ki}} p_i^+ / p_i^-, & \text{if } b_{ki} \leq 0; \end{cases}$$

We write $(\mathbf{x}', \mathbf{p}', B') = \mu_k((\mathbf{x}, \mathbf{p}, B))$. Notice that $\mu_k(\mu_k((\mathbf{x}, \mathbf{p}, B))) = (\mathbf{x}, \mathbf{p}, B)$. We say that two seeds are *mutation-equivalent* if one is obtained from the other by a sequence of seed mutations. Similarly we say that two clusters or two exchange matrices are *mutation-equivalent*.

For any skew-symmetrizable matrix B we define *initial seed*

$$(\mathbf{x}, \mathbf{p}, B) = (\{x_1, \dots, x_n\}, \{p_1^\pm, \dots, p_n^\pm\}, B),$$

where B is the *initial exchange matrix*, $\mathbf{x} = \{x_1, \dots, x_n\}$ is the *initial cluster*, $\mathbf{p} = \{p_1^\pm, \dots, p_n^\pm\}$ is the *initial coefficient tuple*.

Cluster algebra $\mathcal{A}(B)$ associated with the skew-symmetrizable $n \times n$ matrix B is a subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by all cluster variables of the clusters mutation-equivalent to the initial seed (\mathbf{x}, B) .

Cluster algebra $\mathcal{A}(B)$ is called *of finite type* if it contains only finitely many cluster variables. In other words, all clusters mutation-equivalent to initial cluster contain totally only finitely many distinct cluster variables.

Two most important properties of cluster algebra are Laurent phenomenon [15] and finite type classification [16]. More exactly, Laurent phenomenon states that any cluster variable is expressed as a Laurent polynomial in terms of the initial cluster. The remarkable finite type classification claims that cluster algebras of finite type are in one-to-one correspondence with the Dynkin diagrams of finite type.

2.1. Generalized cluster transformations. Now we introduce more general cluster transformations.

Assume that B is a skew-symmetrizable integer matrix such that all elements in its k th row are divisible by d_k . Define $\beta_{kj} = b_{kj}/d_k$.

Lemma 2.3. Let $B' = \mu_l(B)$ is obtained from B by mutation in direction l . Then, all entries b'_{kj} of k th row of B' are divisible by d_k .

Proof. The statement follows immediately from matrix mutation 2.1. \square

Let us fix d_k for all k from 1 to n and we assume that all elements b_{kj} of k th row of integer skew-symmetrizable matrix B are divisible by d_k .

For a collection $\mathbf{p}_i = (p_{i;0}, \dots, p_{i;d_i})$ we define the polynomial $\theta_i[\mathbf{p}_i](u, v) = \sum_{\ell=0}^{d_i} p_{i;\ell} u^\ell v^{d_i-\ell}$ be a polynomial of degree d_i . The corresponding inhomogeneous polynomial we denote by $\rho_i[\mathbf{p}_i] = \rho_i[\mathbf{p}_i](t) = \sum_{\ell=0}^{d_i} p_{i;\ell} t^\ell$. Note that, $\theta_i[\mathbf{p}_i](u, v) = u^{d_i} \rho_i[\mathbf{p}_i](v/u)$.

We define a generalized seed q of a generalized cluster algebra as a triple $q = (\mathbf{x}(q), \overline{\mathbf{p}(q)}, B(q))$, where $\mathbf{x} = (x_1(q), \dots, x_n(q))$ is a n -tuple of cluster variables in seed q , $\mathbf{p}(q) = (\mathbf{p}_1(q), \dots, \mathbf{p}_n(q))$ is n -tuple of coefficient collections $\mathbf{p}_i(q) = (p_{i;0}(q), \dots, p_{i;d_i}(q))$ is a $d_i + 1$ -tuple of coefficients of $\theta_i[q]$, and, finally, $B(q)$ is an exchange $n \times n$ matrix.

Generalized cluster mutations are described by the following formulas:

Exchange matrix is mutated in direction k by the rule 2.1. Introduce $u_{j;>0} = \prod_{\beta_{j,\ell}>0} x_\ell^{\beta_\ell}$, $u_{j;<0} = \prod_{\beta_{j,\ell}<0} x_\ell^{-\beta_\ell}$, Mutation of cluster variables is given by the rule $(\{x'_i\}, \{\mathbf{p}'_i\}, B') = \mu_k(\{x_i\}, \{\mathbf{p}_i\}, B)$:

$$(2.5) \quad x'_i = \begin{cases} x_i, & \text{if } i \neq k; \\ \frac{\theta_k(u_{k;>0}, u_{k;<0})}{x_k}, & \text{otherwise.} \end{cases}$$

Coefficients mutate by the following generalized rule:

$$(2.6) \quad p'_{k;\ell} = p_{k;d_k-\ell}$$

$$(2.7) \quad \text{for } i \neq k \quad p'_{i;j}/p'_{i;r} = \begin{cases} (p_{k;0})^{b_{ki}} p_{i;r}/p_{i;j}, & \text{if } b_{ki} \geq 0; \\ (p_{k;d_k})^{b_{ki}} p_{i;r}/p_{i;j}, & \text{if } b_{ki} \leq 0; \end{cases}$$

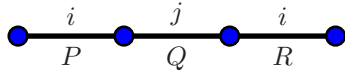
Remark 2.4. Note that if we assume that coefficients of all θ_i do not change under mutation then the corresponding inhomogeneous polynomial ρ_i is reciprocal of degree d_i , i.e., $t^{d_i} \rho(1/t) = \rho(t)$.

Theorem 2.5. (*Laurent property*) Any generalized cluster variable is a Laurent polynomial in initial cluster variables x_i .

Proof. The proof uses the "caterpillar lemma" [15].

Lemma 2.6. Assume that a generalized exchange pattern on $T_{n,m}$ satisfies the following conditions:

- (1) For any edge the polynomial P does not depend on x_k and is not divisible by any x_i , $i \in [n]$.
- (2) Each exchange polynomial has nonnegative coefficients
- (3) For any three edges labeled by i, j, i



we have $L \cdot Q_0^b \cdot P = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$, where b is a negative integer, $Q_0 = Q|_{x_i=0}$, and L is a Laurent monomial whose coefficient lies in \mathbb{A} and is coprime with P .

Then each element $x_i(t)$ for $i \in [n]$, $t \in T_{n,m}$ is a Laurent polynomial in $x_1(t_0), \dots, x_n(t_0)$ with coefficients in \mathbb{A} .

By definition of generalized cluster transformation $P = \theta_i[t]$, $Q = \theta_j[t']$, $R = \theta_i[t'']$.

Note that parts (1) and (2) are evidently satisfied by generalized cluster mutations. It remains to proof part (3). note that if x_i is not included into any monomial of Q then generalized mutation with labels i and j commute and the latter mutation is inverse to the former, namely, $x_i(t''') = x_i(t)$.

Consider the case when x_i enters a monomial of Q . For simplicity denote $b_{ji}(t')$ by a .

By our assumptions $a \neq 0$. Moreover, without loss of generality we can assume that $a > 0$, otherwise we replace B by $-B$. Since Q is determined by the homogeneous polynomial of two variables where only one variable contains a positive power of x_i then x_i enters all monomial of Q but one. Hence, $Q_0 = Q|_{x_i=0}$ is a monomial. Moreover, $Q_0 = p_{j;0} \prod_{b_{jk}(t') > 0} x_k^{b_{jk}(t')}$. Note that $P = \theta_i(u_{>0}(t'), u_{<0}(t'))$. By the mutation rule 2.2,

$$b_{il}(t'') = \begin{cases} b_{il}(t'), & \text{if } b_{jl}(t') \leq 0; \\ b_{il}(t') + ab_{jl}(t'), & \text{otherwise.} \end{cases}$$

By the definition of generalized cluster transformation $R = \theta_i(u_{>0}(t''), u_{<0}(t''))$.

For $q = t'$ or t'' introduce $\tau_q = \prod_{\ell} x_{\ell}(q)^{\beta_{i\ell}(q)} = \frac{u_{>0}(q)}{u_{<0}(q)}$.

Finally,

$$\begin{aligned} \tau_{t''}|_{x_j \leftarrow \frac{Q_0}{x_j}} &= \prod_{l \neq j} x_l^{\beta_{il}(t')} \cdot \left(\frac{Q_0}{x_j} \right)^{-a/d_i} \prod_{\beta_{jl}(t') > 0} x_l^{a\beta_{jl}(t')} = \\ &= \prod_l x_l^{\beta_{il}(t)} \left(p_{j;0} \prod_{b_{jl}(t') > 0} x_l^{b_{jl}(t')} \right)^{-a/d} \left(\prod_{b_{jl}(t') > 0} x_l^{a\beta_{jl}(t')} \right) = p_{j;0}^{-a/d_i} \tau_{t'} = \frac{1}{p_{j;0}^{a/d_i} \tau_t} \end{aligned}$$

It is enough to notice that 2.5 and 2.6 imply that $\rho_{t;i}(\tau_t) = \rho_{t'';i} \left(\tau_{t''}|_{x_j \leftarrow \frac{Q_0}{x_j}} \right)$.

Therefore, $R|_{x_j \leftarrow \frac{Q_0}{x_j}} = P \cdot \hat{L}$, where \hat{L} is a Laurent monomial. \square

Theorem 2.7. *Generalized cluster algebras of finite type satisfy the same Cartan-Killing classification as the standard cluster algebras.*

Proof. The proof repeats the one of [16]. The only differences make the proofs of the fact that the only finite type generalized cluster algebras of rank two correspond to A_2, B_2, G_2 types. It is checked by direct computation similar to one in [16]. Note first that in A_2 -case formulas for generalized cluster transformation coincide with formulas for the standard cluster transformation. In B_2 -case the polynomial degrees of theta-polynomials are two and one. Set the theta polynomials in the initial cluster $\theta_1(u, v) = au^2 + buv + cv^2$, $\theta_2(u, v) = pu + qv$. Then, we immediately obtain $(x, y) \xleftarrow{\mu_1} (\mu_1)(x_1, y) \xleftarrow{\mu_2} (x_1, y_1) \xleftarrow{\mu_1} (x_2, y_1) \xleftarrow{\mu_2} (x_2, y_2) \xleftarrow{\mu_1} (x, y_2) \xleftarrow{\mu_2} (x, y)$, where

$$\begin{aligned} x_1 &= (a + by + cy^2)/x, \\ y_1 &= (px + qa + bqy + cxy^2)/xy, \\ x_2 &= (a^2q^2 + 2apqx + acq^2y^2 + abq^2y + bpqxy + p^2x^2)/xy^2, \\ y_2 &= (qa + px)/y. \end{aligned}$$

Similar computations lead to the 8-cycle in G_2 -case. Note that degrees of polynomials $\deg(\theta_1) = 3$, $\deg(\theta_2) = 1$. We set $\theta_1(u, v) = au^3 + bu^2v + cuv^2 + dv^3$, $\theta_2(u, v) = pu + qv$. $(x, y) \xleftarrow{\mu_1} (\mu_1)(x_1, y) \xleftarrow{\mu_2} (x_1, y_1) \xleftarrow{\mu_1} (x_2, y_1) \xleftarrow{\mu_2} (x_2, y_2) \xleftarrow{\mu_1} (x_3, y_2) \xleftarrow{\mu_2} (x_3, y_3) \xleftarrow{\mu_1} (x, y_3) \xleftarrow{\mu_2} (x, y)$, where

$$\begin{aligned} x_1 &= (a + by + cy^2 + dy^3)/x, \\ y_1 &= (px + aq + bqy + cqy^2 + dqy^3)/xy, \\ x_2 &= (a^3q^3 + 2a^2cq^3y^2 + 2a^2dq^3y^3 + 3a^2pq^2x + 2a^2bq^3y + 2abdq^3y^4 + 3ap^2qx^2 + 4abpq^2xy + ac^2q^3y^4 + 2acdq^3y^5 + ad^2q^3y^6 + 3acpq^2xy^2 + 3adpq^2xy^3 + ab^2q^3y^2 + 2abcq^3y^3 + bcpq^2xy^3 + bdpq^2xy^4 + p^3x^3 + b^2pq^2xy^2 + 2bp^2qx^2y + p^2cq^2y^2)/x^2y^3, \\ y_2 &= (q^2a^2 + abq^2y + acq^2y^2 + adq^2y^3 + 2apqx + bpqxy + p^2x^2)/xy^2, \\ x_3 &= (a^2dq^3y^3 + a^2cq^3y^2 + acpq^2xy^2 + a^2bq^3y + 2abpq^2xy + bp^2qx^2y + a^3q^3 + 3a^2pq^2x + 3ap^2qx^2 + p^3x^3)/xy^3, \\ y_3 &= (px + aq)/y. \end{aligned}$$

□

We note that according to [18] the generalized cluster transformation preserves presymplectic structure compatible with cluster algebra structure. Similarly, the secondary generalized cluster transformation preserves the compatible Poisson bracket.

Theorem 2.8. ([18]) *Poisson structure compatible with a cluster algebra is compatible with the corresponding generalized cluster transformations.*

In the next section we describe generalized cluster structure associated with triangulated surfaces with orbifold points.

3. TEICHMÜLLER SPACE OF SURFACES WITH HOLES AND ORBIFOLD POINTS OF ARBITRARY ORDER

We now demonstrate how the above mutations with reciprocal polynomials of the second order appear in the description of Teichmüller spaces of Riemann surfaces of arbitrary genus with nonzero number of holes (punctures) and with an arbitrary number r of orbifold points of arbitrary orders.

3.1. The ideal triangle decompositions of orbifold Riemann surfaces and cluster variables. We now present the geometric pattern underlying the algebraic construction of cluster variables corresponding to orbifold Riemann surfaces.

Particular cases of Riemann surfaces with orbifold points of order 2 and 3 are discussed in [2, 4], and [6]. For relation between skew-symmetrizable cluster algebras of finite mutation type and surfaces with orbifold points see [13].

We consider a regular Riemann surface $\Sigma_{g,s,r}$ of genus g with $s > 0$ holes and with a number $r \geq 0$ of orbifold points; orders of these points p_i , $i = 1, \dots, r$ are positive integers greater than one.

We introduce the marking on the set of orbifold points splitting this set into nonintersecting subsets δ_k (maybe empty), $k = 1, \dots, s$, $\sum_{k=1}^s |\delta_k| = r$. We then assign the subset δ_k to the k th hole for every k and introduce a cyclic ordering inside each subset δ_k .

To construct the generalization of the ideal triangle decomposition [8, 22], we first remove from the surface all the hyperbolic domains of holes bounded by the corresponding perimeter geodesic lines (with their closures, which are these perimeter lines). Second, we choose for each orbifold point from the set δ_k a domain containing this point and bounded by a geodesic curve whose both ends spiral to the k th

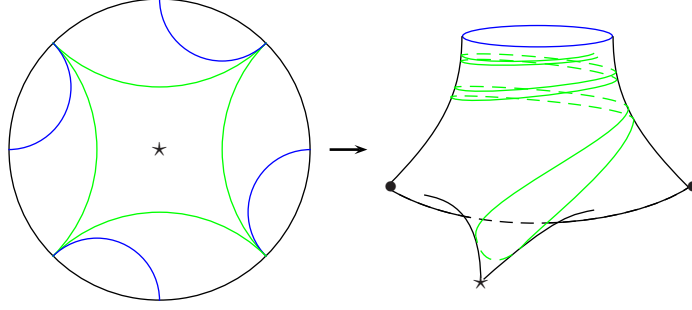


FIGURE 1. The right part represents an example of a regular genus zero Riemann surface with a \mathbb{Z}_4 -orbifold point marked by a star; the bounding geodesic line is the image of a side of an ideal equilateral square in the Poincaré disc depicted in the left part. Both ends of the geodesic line spiral asymptotically to the closed geodesic that is the boundary of a hole.

hole as shown in Fig. 1.¹ We remove from the remaining part of the surface all such domains. The remaining part of $\Sigma_{g,s,r}$ admits splitting into ideal triangles; a copy of this splitting can be drawn as a connected ideal polygon in the Poincaré disc; the sides of this polygon are of two sorts: those that are not pre-images of geodesics going around orbifold points must be pairwise identified; to a side that is a pre-image of the geodesics going around \mathbb{Z}_p orbifold point we attach (from outside) an equilateral ideal p -gone with the orbifold point situated at its geodesic center; this p -gone is the p -fold covering of the removed domain enclosing the orbifold point.

An example of a fundamental domain of the Riemann surface $\Sigma_{1,1,2}$ of genus one with one hole and with two \mathbb{Z}_3 orbifold points is in Fig. 2.

We use the standard geometric correspondence between cluster variables and λ -lengths: at each point at the absolute that is a vertex of an ideal triangle we set an horocycle; the λ -length ℓ_a is then the geodesic length of the part of side a enclosed between two horocycles based at its endpoints. The correspondence reads:

$$(3.8) \quad a = e^{\ell_a/2},$$

where ℓ_a is understood as the signed distance between horocycles: ℓ_a is negative when the corresponding horocycles overlap. Here a is the cluster variable associated with the edge.

For an ideal quadrangle with the (cyclically enumerated) sides a_i , $i = 1, \dots, 4$ and with diagonals d and d' we have the standard two-term cluster relation

$$(3.9) \quad dd' = a_1a_3 + a_2a_4.$$

We now consider mutations for cluster variables of the ideal p -gone corresponding to a \mathbb{Z}_p -orbifold point. We consider the pattern in the left-hand side of Fig. 3 and perform a sequence of mutations (3.9) to come to the pattern in the right-hand

¹In the case of a \mathbb{Z}_2 -orbifold point, this domain has zero area because the corresponding geodesics goes directly to the orbifold point, reflects at it, and repeats its path in the opposite direction.

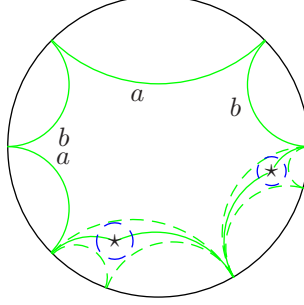


FIGURE 2. The Poincaré disc with depicted fundamental domain for the genus-one surface $\Sigma_{1,1,2}$ with one hole and two \mathbb{Z}_3 orbifold points marked by \star). Solid lines constitute the boundary of a fundamental domain and dashed lines are sides of the related ideal triangles. The sides with labels a and b are pairwise identified. Inscribed circles with centers at the \mathbb{Z}_3 orbifold points all have the radius $\frac{1}{2} \log 3$.

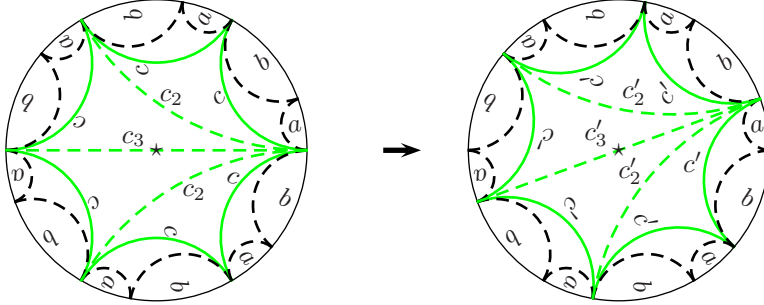


FIGURE 3. The mutation in the ideal p -gone (here $p = 6$), which is the p -fold covering of the domain around a \mathbb{Z}_p -orbifold point (marked by \star). We perform standard 2-term mutations (3.9) on the set of cluster variables to come from the pattern in the left-hand side to the one in the right-hand side.

side. For an equilateral p -gone, the cluster variables c_k for k -diagonal are

$$(3.10) \quad c_k = c \frac{\sin(\pi k/p)}{\sin(\pi/p)}, \quad (c_1 = c).$$

The easy combinatorics then yields

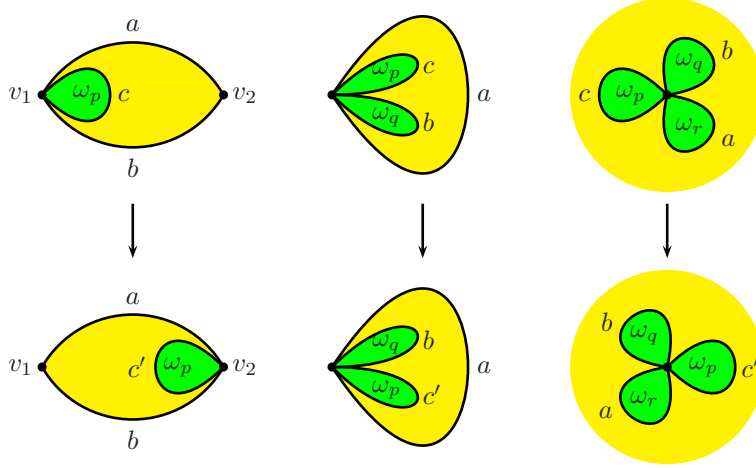
Lemma 3.1. *λ -lengths satisfy the following relation*

$$cc' = a^2 + 2 \cos(\pi/p)ab + b^2.$$

Remark 3.2. Relation in Lemma 3.1 is a generalized cluster mutation described in Sec. 2. Note that since we use only two-term transformations 3.9 to prove

Lemma 3.1, the positivity property for generalized mutations of such form follows from the one for the standard mutations.

To simplify the description, it is convenient to introduce the notion of *petal surface*. Petals are the domains containing orbifold points that were removed on the second step of constructing the ideal triangulation. To the k th hole, we associate the bouquet of $|\delta_k|$ petals (with no petals if δ_k is empty), each petal carries, besides its cluster variable, the number $\omega_p = 2\cos(\pi/p)$. The mutation then occurs inside the corresponding ideal triangle (painted by a light color), and we have three cases depending on whether the adjacent sides are petals themselves:



In all the three cases above, the mutation law is given by Lemma 3.1. In the first case, we transfer the cluster variable from one set δ_k to another set (if the vertices v_1 and v_2 are distinct); these transformations may also change the cyclic ordering of orbifold points inside a set δ_k . Note that the label ω_p remains assigned to the transformed edge.

3.2. Fat graph description for Riemann surfaces with holes and with \mathbb{Z}_p orbifold points. In this subsection and in the rest of the paper, we use the graphs dual to the above ideal triangle decompositions of Riemann surfaces. These graphs are especially useful when describing the Fuchsian groups $\Delta_{g,s,r}$ of Riemann surfaces $\Sigma_{g,s,r}$ and the corresponding geodesic functions.

Definition 3.3. We call a fat graph (a graph with the prescribed cyclic ordering of edges entering each vertex) $\Gamma_{g,s,r}$ a *spine of the Riemann surface* $\Sigma_{g,s,r}$ with g handles, $s > 0$ holes, and r orbifold points of the corresponding orders p_i , $i = 1, \dots, r$, if

- (a) this graph can be embedded without self-intersections in $\Sigma_{g,s,r}$;
- (b) all vertices of $\Gamma_{g,s,r}$ are three-valent except exactly r one-valent vertices (endpoints of “pending” edges), which are placed at the corresponding orbifold points;
- (c) for an orbifold point from the set δ_k , the corresponding pending edge protrudes towards the interior of the face of the graph containing the k th hole; the cyclic ordering of pending edges pointing towards the interior of this face coincide with that of orbifold points in the set δ_k ;

- (d) upon cutting along all edges of $\Gamma_{g,s,r}$ the Riemann surface $\Sigma_{g,s,r}$ splits into s polygons each containing exactly one hole and being simply connected upon gluing this hole.

Edges of this graph are labeled by distinct integers $\alpha = 1, 2, \dots, 6g - 6 + 3s + 2r$, and to each edge we set into the correspondence the real number Z_α .

The first homotopy groups $\pi_1(\Sigma_{g,s,r})$ and $\pi_1(\Gamma_{g,s,r})$ coincide because each closed path in $\Sigma_{g,s,r}$ can be homotopically transformed to a closed path in $\Gamma_{g,s,r}$ (taking into account paths that go around orbifold points) in a unique way. The standard statement in hyperbolic geometry is that conjugacy classes of elements of a Fuchsian group $\Delta_{g,s,r}$ are in the 1-1 correspondence with homotopy classes of closed paths in the Riemann surface $\Sigma_{g,s,r} = \mathbb{H}_+^2 / \Delta_{g,s,r}$ and that the “actual” length ℓ_γ of a hyperbolic element $\gamma \in \Delta_{g,s,r}$ coincides with the minimum length of curves from the corresponding homotopy class; it is then the length of a unique closed geodesic line belonging to this class.

When orbifold points are present, the Fuchsian group contains besides hyperbolic elements also elliptic elements corresponding to rotations about these orbifold points. The corresponding generators \tilde{F}_i , $i = 1, \dots, r$, of the rotations through $2\pi/p_i$ are conjugates of the matrices

$$(3.11) \quad \tilde{F}_i = U_i F_{p_i} U_i^{-1}, \quad F_p = \begin{pmatrix} 0 & 1 \\ -1 & -w \end{pmatrix}, \quad w = 2 \cos \pi/p, \quad p \geq 2.$$

The real numbers Z_α in Definition 3.3 are the h -lengths [22]: they are called the (*Thurston*) *shear coordinates* [23],[1] in the case of punctured Riemann surface (without boundary components). We preserve this notation and this term also in the case of orbifold surfaces. These coordinates are related to the cross-ratio relation for two adjacent ideal triangles constituting the ideal quadrangle with the respective vertices (in the cyclic order) a, b, c, d and diagonal bd . At the same time, they are related to the cluster variables a_i corresponding to the sides of the corresponding quadrangle. We have

$$(3.12) \quad e^Z = -\frac{(b-c)(d-a)}{(b-a)(d-c)} = \frac{a_1 a_3}{a_2 a_4},$$

and we obtain the parameter Z_α choosing $\{a, b, c, d\} = \{-1, 0, e^{Z_\alpha}, \infty\}$.

For example, the fat graph corresponding to the pattern in Fig. 2 is depicted in Fig. 4.

3.3. The Fuchsian group $\Delta_{g,s,r}$ and geodesic functions. We now describe combinatorially the conjugacy classes of the Fuchsian group $\Delta_{g,s,r}$. Every time the path homeomorphic to a (closed) geodesic γ goes through the edge with the label α we insert [8] the matrix of the Möbius transformation

$$(3.13) \quad X_{Z_\alpha} = \begin{pmatrix} 0 & -e^{Z_\alpha/2} \\ e^{-Z_\alpha/2} & 0 \end{pmatrix}.$$

We also have the “right” and “left” turn matrices to be set in proper places when a path makes the corresponding turns at three-valent vertices,

$$(3.14) \quad R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

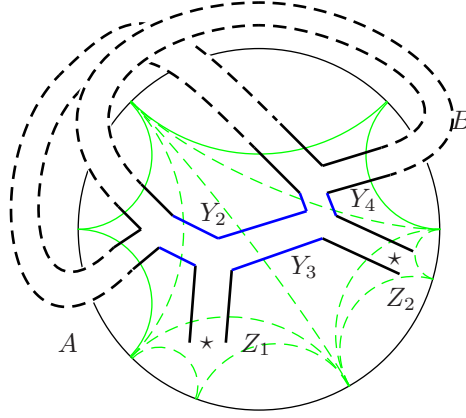


FIGURE 4. The fat graph corresponding to the ideal triangle partition of the fundamental domain in Fig. 2. The real numbers Z_1 and Z_2 are associated to the pending edges and A , B , and Y_2 , Y_3 , and Y_4 to the inner edges.

New elements of the Fuchsian group correspond to rotations of geodesic lines when going around orbifold points indicated by star-vertices; for a \mathbb{Z}_p orbifold point we then insert the matrix F_p (3.11) into the corresponding string of 2×2 -matrices (when we go around the orbifold point counterclockwise as in Fig. 5(a)). When the order of the orbifold point is larger than two, we can go around it k times; *we then have to insert the matrix $(-1)^{k+1}F_p^k$ into the product of 2×2 -matrices*. For example, parts of geodesic functions in the three cases in Fig. 5 read

$$(3.15) \quad \begin{aligned} (a) & \quad \dots X_X L X_Z F_p X_Z L X_Y \dots, \\ (b) & \quad \dots X_X L X_Z (-F_p^2) X_Z R X_X \dots, \\ (c) & \quad \dots X_Y R X_Z (F_p^3) X_Z L X_Y \dots \end{aligned}$$

Note that $F_p^p = (-1)^{p-1}\mathbb{E}$, so going around the \mathbb{Z}_p orbifold point p times merely corresponds to avoiding this orbifold point due to the simple equality (note that $X_S^2 = -\mathbb{E}$ and $L^2 = -R$)

$$X_X L X_Z (-1)^{p-1} F_p^p X_Z L X_Y = X_X L X_Z^2 L X_Y = -X_X L^2 X_Y = X_X R X_Y.$$

(For the \mathbb{Z}_2 orbifold points this pattern was first proposed by Fock and Goncharov [10]; the graph morphisms were described in [4].)

An element of a Fuchsian group has then the typical structure

$$(3.16) \quad P_\gamma = L X_{Y_n} R X_{Y_{n-1}} \dots R X_{Y_2} L X_{Z_1} (-1)^{k+1} F_p^k X_{Z_1} R X_{Y_1},$$

where Y_i are variables of “internal” edges and Z_j are those of pending edges. The corresponding *geodesic function*

$$(3.17) \quad G_\gamma \equiv \text{Tr } P_\gamma = 2 \cosh(\ell_\gamma/2)$$

is expressed via the actual length ℓ_γ of the closed geodesic on the Riemann surface.

Remark 3.4. Note that the combinations

$$R X_y = \begin{pmatrix} e^{-Y/2} & -e^{Y/2} \\ 0 & e^{Y/2} \end{pmatrix} \quad \text{and} \quad L X_y = \begin{pmatrix} e^{-Y/2} & 0 \\ -e^{-Y/2} & e^{Y/2} \end{pmatrix}$$

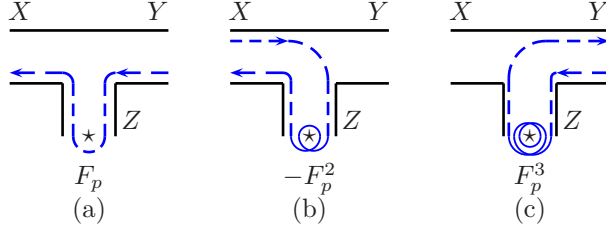


FIGURE 5. Part of a graph with a pending edge. Its endpoint with the orbifold point is directed toward the interior of the boundary component this point is associated with. The variable Z corresponds to the respective pending edge. We present four typical examples of geodesics undergoing single (a), double (b), and triple (c) rotations at the \mathbb{Z}_p orbifold point.

as well as products of any number of these matrices have the sign structure $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$, so the trace of any of P_γ in the absence of orbifold points is a sum of exponentials with positive integer coefficients; this sum always include the terms $e^{Y_1/2+\dots+Y_n/2}$ and $e^{-Y_1/2-\dots-Y_n/2}$ being therefore always greater or equal two thus describing a hyperbolic or parabolic element; the latter is possible only if $Y_1 + \dots + Y_n = 0$ and only if the turn matrices in (3.16) are all either R or L , which corresponds to a path going along the boundary of a face; all such passes are homeomorphic to the hole boundaries, and the condition that the sum of Y_i equals zero indicates the degeneration of a hole into a puncture.

The group generated by elements (3.11) together with translations along A - and B -cycles and around holes not necessarily produces a regular (metrizable) surface because its action is not necessarily discrete. We formulate the necessary and sufficient conditions for producing a *regular* surface in terms of graphs (see [3] for the \mathbb{Z}_2 orbifold point case).²

To formulate the regularity condition, we interpret passages around orbifold points as paths in the p -fold covering of the geodesic neighborhood in Fig. 1. For this, we take the subgraph in Fig. 6 dual to the ideal triangle decomposition in Fig. 3. When splitting the equilateral p -gone into ideal triangles we break the p -fold symmetry, so now the shear coordinates Z_i , $i = 1, \dots, p$, on the fat graph edges dual to the corresponding p -gone sides and Y_j , $j = 2, \dots, p-2$, on the edges dual to the diagonals of the p -gone are different.

We first identify the parameter Z in (3.15) to be

$$(3.18) \quad e^Z = \frac{\sin(\pi/p - \phi/2)}{\sin(\phi/2)},$$

We can then derive the explicit relations between the parameter Z in (3.18) and the variables Z_i and Y_j determined by the standard cross-ratio relations (3.12). The vertices of the ideal p -gone are situated at the points $e^{i2\pi k/p}$, $k = 0, \dots, p-1$, and

²In what follows, we call a Riemann surface regular if it is locally a smooth constant-curvature surface everywhere except exactly r orbifold points.

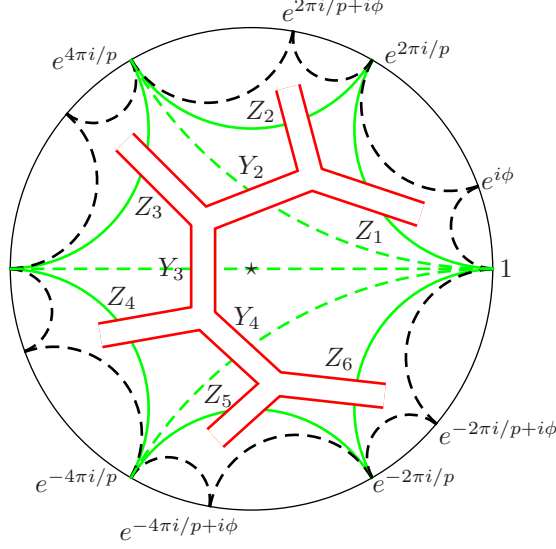


FIGURE 6. The (tree-like) subgraph dual to the ideal triangle decomposition of the ideal equilateral p -gone in Fig. 3. All the variables Z_α and Y_β are determined by the cross-ratio relations in the corresponding ideal quadrangles based on the points from the sets $\{e^{2\pi ik/p}\}$ and $\{e^{2\pi ik/p+i\phi}\}$, $k = 0, \dots, p-1$.

p copies of the vertex of an additional ideal triangle adjacent to the p -gone side are $e^{i\phi+2\pi ik/p}$, $k = 0, \dots, p-1$.

Using the cross-ratio relations to calculate Z_i and Y_j we find the exact relations between these variables and the variable Z given by (3.18):

$$\begin{aligned}
 e^{Z_1} &= e^Z \frac{\sin(2\pi/p)}{\sin(\pi/p)}, \\
 e^{Z_p} &= e^Z \frac{\sin(\pi/p)}{\sin(2\pi/p)}, \\
 (3.19) \quad e^{Z_k} &= e^Z \frac{\sin((k-1)\pi/p)}{\sin(k\pi/p)}, \quad k = 2, \dots, p-1,
 \end{aligned}$$

$$(3.20) \quad e^{Y_k} = \frac{\sin((k+1)\pi/p)}{\sin((k-1)\pi/p)}, \quad k = 2, \dots, p-2.$$

The following 2×2 -matrix equalities can be verified directly:

$$(3.21) \quad X_Z F_p X_Z = X_{Z_1} L X_{Z_2} = X_{Z_k} L X_{Y_k} L X_{Z_{k+1}} = X_{Z_{p-1}} L X_{Z_p}, \quad k = 2, \dots, p-2.$$

We then have the following lemma.

Lemma 3.5. *We have the following explicit 2×2 -matrix relations for the shear variables in the equilateral p -gone in Fig. 6 given by (3.19) and (3.20):*

$$X_Z F_p X_Z = X_{Z_1} L X_{Z_2}$$

$$\begin{aligned}
(3.22) \quad X_Z(-F_p^2)X_Z &= X_{Z_1}RX_{Y_2}LX_{Z_3} \\
&\vdots \\
X_Z(-1)^{k-1}F_p^kX_Z &= X_{Z_1}RX_{Y_2}R \cdots RX_{Y_k}LX_{Z_{k+1}}, \quad k = 2, \dots, p-2 \\
X_Z(-1)^pF_p^{p-1}X_Z &= X_{Z_1}RX_{Y_2}R \cdots RX_{Y_{p-2}}RX_{Z_p}.
\end{aligned}$$

The *proof* uses equalities from (3.21) for constructing the longer chain using that $X_S X_S = -\mathbb{E}$ for any variable S and that $L^2 = -R$. For example, we obtain the r.h.s. of the second equality in (3.22) multiplying $X_{Z_1}LX_{Z_2} \cdot X_{Z_2}LX_{Y_2}LX_{Z_3}$ whereas the l.h.s. merely becomes $X_Z F_p X_Z \cdot X_Z F_p X_Z = X_Z(-1)F_p^2 X_Z$. All other equalities are obtained if we continue this chain of multiplications.

Due to Lemma 3.5, all “rotations” about orbifold points $X_Z(-1)^{k+1}F_p^k X_Z$ are now presented as the standard products of matrices X_S (with real S) alternated with the matrices of left and right turns (3.14), which means that the conclusion of Remark 3.4 is valid in this case as well: as soon as in the original spine $\Gamma_{g,s,r}$ all the parameters Z_α of pending edges are real, all the geodesic functions constructed on a Riemann surface with orbifold points are Laurent polynomials with *positive integer coefficients* of the “new” real variables $Z_i^{(\alpha,p)}$, $Y_j^{(p)}$ (where the superscripts α, p indicate that these variables are completely determined by the original variable Z_α and the order p of the orbifold point) and “old” variables of “internal” edges of the spine $\Gamma_{g,s,r}$, corresponding to usual partitions into ideal triangles. So, again, in the trace of every product of form (3.16) we necessarily have the term $2 \cosh(\sum_{\beta=1}^n X_\beta)$, where the sum ranges all edges (new and internal ones) the corresponding path goes over, and we let X denote the variables on all these edges disregarding their origins. Every such trace is therefore a positive number greater or equal two, and the corresponding element of the group will be either hyperbolic or parabolic (the latter is possible only for geodesics around holes and only if a hole reduces to a puncture). The only elliptic elements are precisely conjugates of F_p^k . We therefore come to the theorem

Theorem 3.6. *We have a metrizable Riemann surface for any choice of real numbers Z_α associated to the edges of an original spine $\Gamma_{g,s,r}$. The converse statement is also true: for any metrizable Riemann surface $\Sigma_{g,s,r}$ we have a spine $\Gamma_{g,s,r}$ with real numbers associated to its edges such that the lengths of geodesics on $\Sigma_{g,s,r}$ are given by traces of products (3.16) corresponding to paths in the spine.*

The *proof* of the second statement was performed in [4] for \mathbb{Z}_2 -orbifold points. It is based on the (obvious) existence of the ideal triangle decomposition described in Sec. 3.1 for any metrizable Riemann surface and can be straightforwardly generalized to the case of orbifold points of any type. We have therefore parameterized all possible regular surfaces in terms of the $(6g - 6 + 3s + 2r)$ -tuple of real coordinates $\{Z_\alpha\}$.

Corollary 3.7. *The decorated Teichmüller space $\mathfrak{T}_{g,s,r}^H$ of Riemann surfaces with holes and orbifold points is the space $\mathbb{R}^{6g-6+3s+2r}$ of real parameters on the edges of a spine $\Gamma_{g,s,r}$.*

4. MAPPING CLASS GROUP TRANSFORMATIONS

4.1. Poisson structure. One of the most attractive properties of the graph description is a very simple Poisson algebra on the set of parameters Z_α . The following

result is the straightforward generalization of the theorem formulated for surfaces without marked points in [8] and for surfaces with order-2 orbifold points in [10] (see also [2]).

Theorem 4.1. *In the coordinates Z_α on any fixed spine corresponding to a surface with orbifold points, the Weil–Petersson bracket B_{WP} is given by*

$$(4.23) \quad \{f(\mathbf{Z}), g(\mathbf{Z})\} = \sum_{\substack{\text{3-valent} \\ \text{vertices } \alpha = 1}}^{4g+2s+|\delta|-4} \sum_{i=1}^{3 \bmod 3} \left(\frac{\partial f}{\partial Z_{\alpha_i}} \frac{\partial g}{\partial Z_{\alpha_{i+1}}} - \frac{\partial g}{\partial Z_{\alpha_i}} \frac{\partial f}{\partial Z_{\alpha_{i+1}}} \right),$$

where the sum ranges all the three-valent vertices of a graph and α_i are the labels of the cyclically (counterclockwise) ordered ($\alpha_4 \equiv \alpha_1$) edges incident to the vertex with the label α irrespectively on whether these edges are internal or pending edges of the graph. This bracket gives rise to the Goldman bracket on the space of geodesic length functions [17].

We identify the exchange matrix B with the matrix of the Poisson relations for the variables Z_α .

The center of this Poisson algebra is provided by the proposition.

Proposition 4.2. *The center of the Poisson algebra (4.23) is generated by elements of the form $\sum Z_\alpha$, where the sum ranges all edges of $\Gamma_{g,s,r}$ belonging to the same boundary component taken with multiplicities. This means, in particular, that each pending edge, irrespectively on the type of orbifold point it corresponds to, contributes twice to such sums. The dimension of this center is obviously s .*

for the proof in the general case see Appendix B of [7].

4.2. Flip morphisms of fat graphs. In this section, we present the complete list of mapping class group transformations that enable us to change numbers $|\delta_k|$ of orbifold points associated with the k th hole, change the cyclic ordering inside any of the sets δ_k , flip any inner edge of the graph and, eventually, change the orientation of the geodesic spiraling to the hole perimeter (in the case where we have more than one hole).³ We can therefore establish a morphism between any two of the graphs belonging to the same class $\Gamma_{g,s,r}$ with the same (unordered) sets of orbifold point orders $\{p_i\}_{i=1}^r$.

4.2.1. Whitehead moves on inner edges. Given a spine Γ of Σ and assuming that the edge α has distinct endpoints, we may produce another spine Γ_α of Σ by contracting and expanding edge α of Γ , the edge labeled Z in Figure 7. This transformation is dual to the mutation (3.9). We say that Γ_α arises from Γ by a *Whitehead move* (or flip) along the edge α . A labeling of edges of the spine Γ implies a natural labeling of edges of the spine Γ_α ; we then obtain a morphism between the spines Γ and Γ_α .

Proposition 4.3. [5] *Setting $\phi(Z) = \log(1 + e^Z)$ and adopting the notation of Fig. 7 for shear coordinates of nearby edges, the effect of a Whitehead move is*

$$(4.24) \quad W_Z : (A, B, C, D, Z) \rightarrow (A + \phi(Z), B - \phi(-Z), C + \phi(Z), D - \phi(-Z), -Z)$$

In the various cases where the edges are not distinct and identifying an edge with its shear coordinate in the obvious notation we have: if $A = C$, then $A' = A + 2\phi(Z)$; if $B = D$, then $B' = B - 2\phi(-Z)$; if $A = B$ (or $C = D$), then $A' = A + Z$ (or

³These transformations are dual to mutations of cluster variables from Sec. 2.

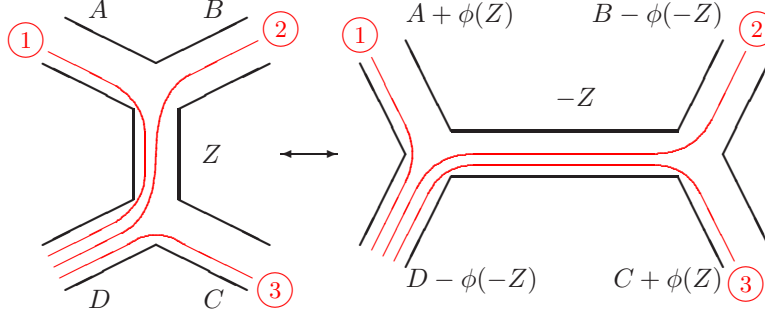


FIGURE 7. Flip, or Whitehead move on the shear coordinates Z_α . The outer edges can be pending, but the edge with respect to which the morphism is performed must be an internal edge. We also indicate the correspondences between geodesic paths under the flip.

$C' = C + Z$); if $A = D$ (or $B = C$), then $A' = A + Z$ (or $B' = B + Z$). Any subset of edges A, B, C , and D can be pending edges of the graph.

We have the lemma establishing the properties of invariance w.r.t. the flip morphisms [5].

Lemma 4.4. *Transformation (4.24) preserves the traces of products over paths (3.17) (the geodesic functions) and transformation (4.24) simultaneously preserves Poisson structure (4.23) on the shear coordinates.*

4.2.2. *Whitehead moves on pending edges.* Choosing other representatives of the orbifold points in the Poincaré disc, we obtain different fundamental domains with different cyclic ordering of the (preimages) of the orbifold points s_i ($i = 1, \dots, |\delta_k|$) possibly with transferring orbifold points from one set δ_k to another set $\delta_{k'}$.

Analogously to the mutation in Fig. 3, flipping the pending edge corresponds to choosing another fundamental domain, as shown in Fig. 8. We take there e^Z given by formula (3.18) and $e^{Y_{1,2}}$ and $e^{\tilde{Y}_{1,2}}$ given by the standard cross-ratio relations, for example,

$$e^{Y_2} = \frac{(1 - e^{i\phi})(e^{ib} - e^{2\pi i/p})}{(e^{i\phi} - e^{ib})(1 - e^{2\pi i/p})}, \quad e^{\tilde{Y}_2} = \frac{(e^{-2\pi i/p+ib} - 1)(e^{-2\pi i/p+i\phi} - e^{i\phi})}{(e^{-2\pi i/p+i\phi} - e^{-2\pi i/p+ib})(1 - e^{i\phi})}.$$

Lemma 4.5. *The transformation in Fig. 8 with e^Z given by (3.18) has the form*

$$(4.25) \quad \{\tilde{Y}_1, \tilde{Y}_2, \tilde{Z}\} = \{Y_1 - \log(1 + we^{-Z} + e^{-2Z}), Y_2 + \log(1 + we^Z + e^{2Z}), -Z\}$$

and is the morphism of the space $\mathcal{T}_{g,s,r}^H$. These morphisms preserve both Poisson structures (4.23) and the geodesic functions. In Fig. 8 any (or both) of Y -variables can be variables of pending edges (the transformation formula is insensitive to it).

Proof. Verifying the preservation of Poisson relations (4.23) is simple, whereas for traces over paths we have four cases, and in each of these cases we have the following 2×2 -matrix equalities to be verified directly:

$$\begin{aligned} X_{Y_2} L X_Z F_p^k (-1)^{k-1} X_Z L X_{Y_1} &= X_{\tilde{Y}_2} R X_{\tilde{Z}} F_p^{p-k+1} (-1)^{p-k} X_{\tilde{Z}} R X_{\tilde{Y}_1}, \\ X_{Y_1} R X_Z F_p^k (-1)^{k-1} X_Z L X_{Y_1} &= X_{\tilde{Y}_1} L X_{\tilde{Z}} F_p^{p-k+1} (-1)^{p-k} X_{\tilde{Z}} R X_{\tilde{Y}_1}, \\ X_{Y_2} L X_Z F_p^k (-1)^{k-1} X_Z R X_{Y_2} &= X_{\tilde{Y}_2} R X_{\tilde{Z}} F_p^{p-k+1} (-1)^{p-k} X_{\tilde{Z}} L X_{\tilde{Y}_2}. \end{aligned}$$

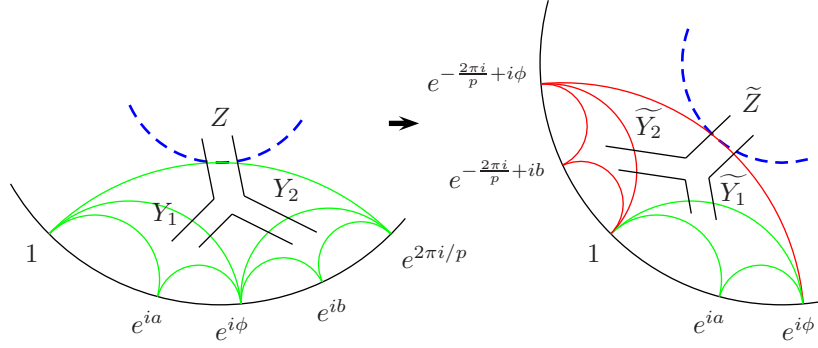


FIGURE 8. The transformation of dual variables (h -lengths) $\{Y_1, Y_2, Z\} \rightarrow \{\tilde{Y}_1, \tilde{Y}_2, \tilde{Z}\}$ described by (4.25) with $w = 2 \cos(\pi/p)$.

Using flip morphisms in Fig. 8 and in formula (4.24), we establish a morphism between any two algebras corresponding to surfaces of the same genus, same number of boundary components, and same numbers of \mathbb{Z}_p -orbifold points of each sort p ; the distribution of latter into the boundary components as well as the cyclic ordering inside each of the boundary component can be arbitrary.

It is a standard tool that if, after a series of morphisms, we come to a graph of the same combinatorial type as the initial one (disregarding labeling of edges but distinguishing between different orbifold types of pending vertices), we associate a *mapping class group* operation to this morphism therefore passing from the groupoid of morphisms to the group of modular transformations.

Remark 4.6. Another way of interpreting transformations (4.25) is as follows. We can imitate the above flips/mutations by introducing a new “hole” with possibly imaginary perimeter P and considering the following chain of *standard* flips:

$$\begin{array}{c}
 \begin{array}{|c|} \hline X \\ \hline \\ \hline Y \\ \hline \end{array}
 \begin{array}{c}
 \text{---} Z \text{---} \\
 \text{---} P \text{---}
 \end{array}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \begin{array}{|c|} \hline X + \log(1+e^Z) \\ \hline \\ \hline Y - \log(1-e^{-Z}) \\ \hline \end{array}
 \begin{array}{c}
 \text{---} Z \text{---} \\
 \text{---} P + Z \text{---}
 \end{array}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \begin{array}{|c|} \hline X + \log(1+e^Z) + \log(1+e^{P+Z}) \\ \hline \\ \hline Y - \log(1-e^{-Z}) - \log(1+e^{-P-Z}) \\ \hline \end{array}
 \begin{array}{c}
 \text{---} Z - P \text{---} \\
 \text{---} P \text{---}
 \end{array}
 \end{array}$$

In this pattern, the variable Z is not *completely* defined by the ideal triangle decomposition: there is an arbitrariness in choosing the “mirror” triangle (we must just choose it consistently). So, in the above sequence of flips, it is useful to shift the variable and consider

$$\bar{Z} = Z + \frac{P}{2}.$$

The transformation for the variables X and Y then just becomes (4.25),

$$\begin{aligned} \begin{bmatrix} X \\ Y \\ \bar{Z} \end{bmatrix} &\rightarrow \begin{bmatrix} X + \log[(1 + e^{\bar{Z}-P/2})(1 + e^{\bar{Z}+P/2})] \\ Y - \log[(1 + e^{-\bar{Z}+P/2})(1 + e^{-\bar{Z}-P/2})] \\ -\bar{Z} \end{bmatrix} = \\ &= \begin{bmatrix} X + \log[1 + \omega_p e^{\bar{Z}} + e^{2\bar{Z}}] \\ Y - \log[1 + \omega_p e^{-\bar{Z}} + e^{-2\bar{Z}}] \\ -\bar{Z} \end{bmatrix}, \end{aligned}$$

where $\omega_p = e^{P/2} + e^{-P/2}$.

Since P is not affected by the above sequence of flips, we can merely erase the corresponding loop and present it as flipping the pending edge, which carries besides the cluster variable \bar{Z} also the “index” variable ω_p , which is not affected by mutations and which is equal to $2\cos(\pi/p)$ in the geometric case.

4.2.3. Changing the spiraling direction. The last mapping class group transformation changes the sign of the hole perimeter:

(4.26)

Lemma 4.7. *Transformation (4.26) preserves the Poisson brackets and the set of geodesic functions.*

Proof. The preservation of the Poisson bracket is obvious because the variable P Poisson commutes with all other variables, whereas the preservation of geodesic functions follows from two matrix equalities:

$$\begin{aligned} X_Y L X_P L X_Y &= X_{Y+P} L X_{-P} L X_{Y+P}, \\ X_Y R X_P R X_Y &= X_{Y+P} R X_{-P} R X_{Y+P}. \end{aligned}$$

We can therefore enlarge the mapping class group of $\mathcal{T}_{g,s,r}^H$ by adding symmetries between sheets of the 2^s -ramified covering of the “genuine” (nondecorated) Teichmüller space $\mathcal{T}_{g,s,r}$.

The geometrical meaning of this transformation is clear: we change the direction of spiraling to the hole perimeter line for all lines of the ideal triangle decomposition that spiral to a given hole like in Fig. 1.

We can summarize as follows.

Theorem 4.8. *The whole mapping class group of $\Sigma_{g,s,r}$ is generated by morphisms described by Lemmas 4.4, 4.5, and 4.7.*

5. CONJECTURES

Lemma 3.1 shows that generalized transformation $a^2 + 2\cos(\pi/p)ab + b^2$ appears as a flip in the presence of an orbifold point of order p . The generalized cluster algebra constructed in this way is a subalgebra of a bigger standard cluster algebra (maybe of infinite rank) associated with triangulated surface while generalized

exchange relation are sequences of standard mutations. Note that the positivity of Laurent polynomials for cluster algebras associated with bordered surfaces is known by [21]. This implies the positivity of Laurent polynomials in generalized cluster algebra associated with triangulations of the surface with arbitrary orbifold points.

We formulate the following conjecture.

Conjecture 5.1. If ρ is a reciprocal polynomial with positive coefficients then any cluster variable of a generalized cluster algebra is expressed as a positive Laurent polynomial in the initial cluster.

We checked by direct inspection that the statement holds for finite type rank 2 cluster algebras.

The example above leads to a natural question:

Question: *Is any generalized cluster algebra a subalgebra of some standard cluster algebra?*

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